18-819F: Introduction to Quantum Computing
47-779/785: Quantum Integer Programming & Quantum Machine Learning

Axioms of Quantum Mechanics

Lecture 09

2021.09.30.

Agenda

- Classical bits
  - Representing the classical bit in a two-dimensional vector space
  - Multiple classical bits as tensor products

- Axioms (rules) of quantum mechanics
  - Quantum bits (qubits)
  - Reversible operations on qubits
  - Unitary operators

Classical Bits

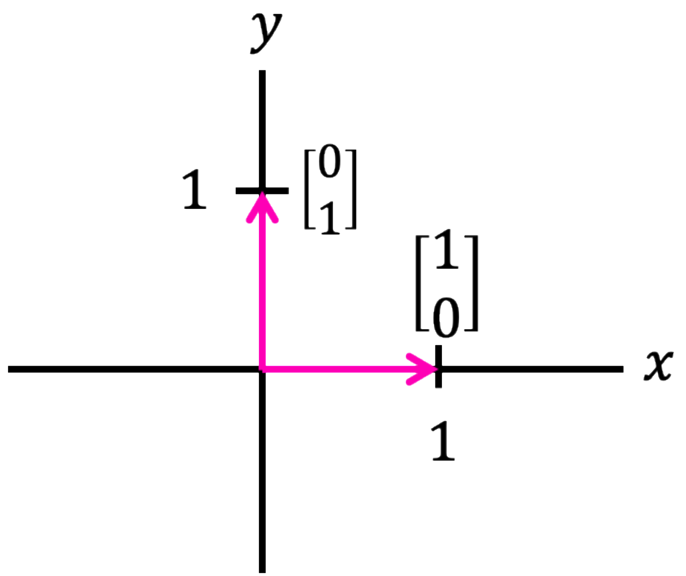
- Classical bits are either 1 or 0 and are represented physically by any device that distinguish between the two states of 1 and 0. A switch is a good example of a physical implementation of a classical bit: the switch is either on (1) or off (0).

- If we have the five classical bits: 10011, we can represent their state by the Dirac ket, $| >$, symbol we have already learned as: $|1>|0>|0>|1>|1>$.

- A pair of classical bits have the following permutations: $|0>|0>, |0>|1>, |1>|0>, |1>|1>$. Since it can be tiresome to write the ket notation for every bit, it is often faster to just write the permutations as: $|00>, |01>, |10>, |11>$, where it is understood that the previous notation is implied.

- Three classical bits have 8 permutations: $|000>, |001>, |010>, |011>, |100>, |101>, |110>, |111>$.

A Useful Perspective of the Classical Bit



- An interesting way to look at the classical bit is to think of it as comprised of two orthogonal unit vectors in a 2-dimensional space as illustrated on the left.

- In the two-dimensional representation, one can think of the $|0>$ and the $|1>$ as the column vectors

$$|0> = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |1> = \begin{bmatrix}0\\1\end{bmatrix} \quad \text{Eqn. (9.1).}$$

- Two classical bits therefore live in a 4-dimensional vector space with the orthonormal basis of

$$|00>, |01>, |10>, |11> \quad \text{Eqn. (9.2).}$$

- If we write the orthonormal basis of (9.2) as a multiplication (as we did on the previous slide), then we can use tensor notation to write

$$|00> = |0>|0> = |0> \otimes |0> \text{ and}$$
$$|01> = |0>|1> = |0> \otimes |1> \ldots \text{and for the rest of (9.3).}$$

Multiple Classical Bits as a Tensor Product

- If we represent 2 classical bits as tensors as we did in (9.3), then a multi-bit representation becomes possible. This hinges on our conception of the classical bit as a vector in two-dimensional space.

- One can not generalize the notion to writing a vector that makes up a multi-bit state composed of the vectors that represent it. Imagine three people, Alice, Bob, and Cathy, have the following classical bits

$$|\text{Alice} > = \binom{a_0}{a_1}, \quad |\text{Bob} > = \binom{b_0}{b_1}, \quad |\text{Cathy} > = \binom{c_0}{c_1}; \quad \text{Eqn. (9.4).}$$

- One can form the multi-bit state of their combined state as

$$|ABC > = \binom{a_0}{a_1} \otimes \binom{b_0}{b_1} \otimes \binom{c_0}{c_1} = \begin{pmatrix} a_0 b_0 c_0 \\ a_0 b_0 c_1 \\ a_0 b_1 c_0 \\ a_0 b_1 c_1 \\ a_1 b_0 c_0 \\ a_1 b_0 c_1 \\ a_1 b_1 c_0 \\ a_1 b_1 c_1 \end{pmatrix} \quad \text{Eqn. (9.5)}$$

...Multiple Classical Bits as a Tensor Product

- The concept expressed in (9.5) can be made more explicit by considering the number 6 and its classical bit representation, where 3 classical bits are used. We have

- $$|6>_3 = |110> = |1>|1>|0> = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}$$ Eqn. (9.6).

- If the 8-component column vector is labeled beginning from the top with 0,1, ... 7, we see that the only component with nonzero value is the 1 in the 7th position, which represents the number 6, which we set out to represent.

Tensor Product Structure of Multiple Bit States

- The tensor product structure of classical multi-bit states can be used in a 2^n-dimensional column vector to represent a state $|n-1>$ for entries that are zero except for the single 1 in the $|n-1>$ position from the top.

- Any integer m in the range $0 \leq m < N$ can be written as one of N orthonormal vectors in an N −dimensional space.

- If $N = 2^n$ and m has a binary representation

$$m = \sum_{i=0}^{n-1} m_i 2^i$$

- Then the column vector for $|m>$ is

$$|m> = |m_{n-1}> \otimes |m_{n-2}> \otimes \cdots \otimes |m_1> \otimes |m_0>.$$

Axioms of Quantum Mechanics

- Quantum mechanics is a description of the building blocks of nature; it is a conceptual framework that best describes atoms, electrons, photons, molecules and other really tiny particles.

- There are a few rules, axioms, or postulates that form the bedrock of this description.

- The axioms are considered incontrovertible truths because there have been ample experiments over the years to verify predictions based on them.

List of the Postulates of Quantum Mechanics

- The 5 main postulates of quantum mechanics as follows:

  - First: The state of a quantum system is described by unit vectors in a complex vector space called Hilbert space. We sometimes also call the state vector a wavefunction, ψ.

  - Second: The probability, P, of finding or measuring a system to be in a certain state or with a certain wavefunction is the square of the modulus of the inner product of the state we measure it to be in and the state that the system originally had before the measurement; put another way, the probability is the square of the modulus of the inner product of the output state, φ, and the input state, ψ; thus $P = |< \varphi|\psi >|^2$.

    a. This axiom expresses Born's rule, which requires that the state vector amplitude or wavefunction amplitude of a quantum system be a *probability amplitude;*

    a. A corollary to this rule is that after measurement, the wavefunction collapses into the measurement basis and all information about the quantum system before the measurement is destroyed.

…List of the Postulates of Quantum Mechanics

- Third: If we want to transform a quantum system, we manipulate its state vector (wavefunction) with operators. Operators transform the state of a system (wavefunction) to other states in the Hilbert space. All quantum mechanical operators are unitary, which means the transpose of the complex conjugate of an operator is equal to its inverse: $(\mathcal{O}^*)^T = \mathcal{O}^\dagger = \mathcal{O}^{-1}$. The axiom is a consequence of the Schrödinger equation which governs temporal evolution.

- Fourth: If we have a quantum system comprised of other systems (super system, so to speak), then the Hilbert space of the composite system is a tensor product of the separate Hilbert spaces. This product is the Kronecker product.

...List of the Postulates of Quantum Mechanics

- Fifth: A measurable physical quantity in quantum mechanics is called an observable and is represented by a Hermitian operator, which (for us) will be a complex square matrix that is the same as its complex conjugate transpose. Observables are represented by eigenvalues of the Hermitian operator.

- We will now attempt to show how the postulates are used in various scenarios.

- Quantum mechanics is a linear theory that follows the rules of linear algebra in a complex vector space called Hilbert space.

Review of Properties of a Basis Vector

- In two-dimensional space, we write a typical 2D vector v as

$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where v_x and v_y are the *coefficients* of the vector in the x and y directions.

- We have explicitly written the *unit* direction vectors as $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;
- These vectors are orthogonal, meaning that $\hat{x}^T . \hat{y} = 0,$ and they are normalized, meaning that $\sqrt{|\hat{x}|} = 1 = \sqrt{|\hat{y}|}$ and are *orthonormal*.

- Orthonormal vectors are *basis vectors* that can be used to express any other vector in the vector space; in the 2D space the basis is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$; other orthonormal basis vectors exist for 2D; this happens to be the simplest.

General Basis Vectors

- It is possible to have $3D$, $4D$, and nD dimensional spaces for vectors and in each of these spaces, one can define or find basis vectors to express any arbitrary vector.

- In addition, vectors and the spaces they are in can be complex and of arbitrary dimension.

- If we have the nD basis $\{v_1, v_2, \ldots v_n\}$, then any vector u in this space can be written as a linear combination of the basis vectors with coefficients c_i which can be complex, thus $u = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$.

- In the *standard basis* a vector u with components $u_1, u_2, \ldots u_n$ can be written as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots u_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

First Postulate: State vectors

- In conventional linear algebra, vectors are mathematical objects with magnitude and direction in 2- or 3-dimensional vector space. We extend the notion of a vector space into spaces with large finite and sometimes infinite dimensions. The only requirement is that vectors continue to be called vectors but are now abstract and could just be a list of things (numbers - which include real and complex numbers). These vectors obey all the rules of linear algebra and add a few that are unique to their abstract nature.

- A vector can therefore be written as: $v = [v_1, v_2, v_3 \dots v_n]$, horizontally or vertically.

- We already defined the classical states $|0 > = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1 > = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as orthogonal unit vectors in 2-dimensional space. They are the unit vector $\hat{x}$ in the x −direction and t $\hat{y}$ in the y−direction.

Unit State Vectors

- In terms of the unit vectors defined in the last slide, consider the state vectors

$$|+> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} \text{ Eqn. (9.7) and}$$

$$|-> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} \text{ Eqn. (9.8);}$$

- What we have done above is to write the given state vector $|+>$ and $|->$ as linear combination (superposition) of the unit state vectors, following a rule from linear algebra; this means we can write

$$|+> = \frac{1}{\sqrt{2}}(|0> +|1>) \text{ and } |-> = \frac{1}{\sqrt{2}}(|0> -|1>) \text{ Eqn. (9.9);}$$

- In general, a vector $|\psi> = a|0> +b|1> = \begin{bmatrix}a\\b\end{bmatrix}$.

Second Postulate: Quantum Measurement

- A system in the state $|\psi> = |-> = \frac{1}{\sqrt{2}}(|0> - |1>)$ can be measure in the orthonormal basis $|u_1> = |0>$ and $|u_2> = |1>$ to get the probability of finding it in the state $|u_1>$ or $|u_2>$; this is an application of the second postulate; thus, probability of finding system in state $|0>$ is

$$P(|0>) = |<0|->|^2 = \left|<0|\frac{1}{\sqrt{2}}(|0> \quad - |1>)\right|^2 = \frac{1}{2}|<0|0> \quad - <0|1>|^2 = \frac{1}{2} \ .$$

- In (9.10), we have used the distributive property, and the orthogonality of unit vectors such that the inner products $<0|0> = 1$ and $<0|1> = 0$;

- In a similar manner, we calculate that $P(|1>) = |<1|->|^2 = 1/2.$

Meaning of Measurement

- Given an orthonormal basis $\{|u_1>, |u_2>, |u_3>, \dots |u_n>\}$ and the quantum system $|\psi>$, we can measure how much of the system state vector is in each of the unit vectors by performing the inner product of each unit vector with the state vector $|\psi>$;

- The probability of measuring how much of the quantum system is in each of the unit vectors can be written as $P(u_i) = |<u_i|\psi>|^2$ Eqn. (9.10);

- Eqn. (9.10) expresses the collapse of the state vector into the unit vector state $|u_i>$;

- We sometimes speak of the projection of the system state vector into the unit state vector $|u_i>$; the projection operator is $P = |u_i><u_i|$ this is what measurement means.

- A quantum measurement is an application of the Born rule.

Calculation of Measurement Probability

- Given a system in the quantum state

$$|\psi> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$$, what is the probability of measuring it in the state $|+>$?

- According to our previous examples, the probability of the measurement is given by performing the following calculation

$$P(+) = |<+|\psi>|^2 = \left|\frac{1}{2}\left[1\ 1\begin{bmatrix}1\\j\end{bmatrix}\right]\right|^2 = \frac{1}{4}|1+j|^2 = \frac{1}{4}(1+j)(1-j) = \frac{1}{2};$$

- Note that in the calculation above we used $|+> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow <+| = \frac{1}{\sqrt{2}}[1\ 1];$
- This example obviates the need for quantum state normalization because for a general state $|\varphi> = \alpha_1|u_1> + \alpha_2|u_2> + \cdots \alpha_n|u_n>$, we want $||\varphi>| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 \ldots |\alpha_n|^2}$; the $|\alpha_i|^2$ are the probabilities of measuring $|\varphi>$ in $|u_i>$; the sum of the probabilities must add to 1.

Third Postulate: Quantum Operations

- Quantum operations transform one state vector to another and as a result, we must insist that the operation not change the length of the vector; in other words, the operators must be unitary;

- In conventional linear algebra, transformations are essentially rotations and rescaling of vectors; the mathematical objects capable of performing rotations and re-scaling of vectors are matrices;
- Common quantum operators (matrices) are:
- $\mathbb{1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is the identity matrix (when multiplied by a scalar λ will perform rescaling); others are $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
- We will next find out what these operators do a state vector; incidentally, these operators are the *Pauli matrices* (yes, that Pauli of the exclusion principle).

Operations with Pauli Matrices

- We will operate on the state $|0> = \begin{bmatrix}1\\0\end{bmatrix}$, which encountered earlier, with the identity matrix, thus

$$I|0> = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0>$$, leaves $|0>$ unchanged as expected;

- Now operate on $|1>$ with the X operator, thus $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0>$, which made the 0 into a 1 and the 1 into a 0; this is looks like negation;

- Operate on the $|+>$ with the $Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ operator, thus

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = |->$$, which transformed $|+>$ to $|->$;

Hadamard Operator

- The Hadamard operator is an important matrix in quantum transformations and is represented by

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \text{ Eqn. (9.10)}$$

- Action of the Hadamard on the states $|1>$, $|0>$, $|+>, |->$ is

$$H|1> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\ 0\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}(|0> \; -|1>) = |-> \; (9.11);$$

$$H|0> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}(|0> \; +|1>) = |+> (9.12);$$

$$H|+> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0> \; (9.13);$$

$$H|-> = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} = |1> \quad (9.14).$$

Constructive and Destructive Interference Operations

- An alternative and revealing way to write the calculation in (9.13) is as follows:

$$H|+> = \frac{1}{\sqrt{2}} H(|0> \ +|1>) = \frac{1}{\sqrt{2}}(H|0> \ +H|1>) = \frac{1}{\sqrt{2}}(|+> + |->) \ \ (9.15);$$

- The last expression in (9.15) can be rewritten as

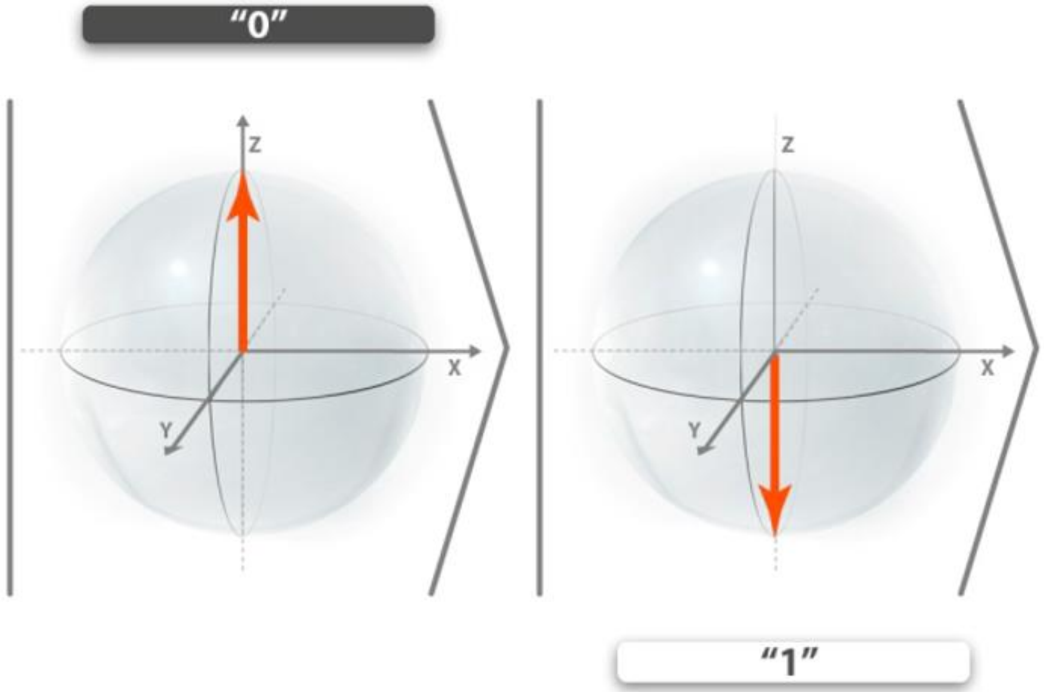
$$\frac{1}{\sqrt{2}}(|+> \ + |->) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0> \ +|1> \ +|0> \ \ -|1>)\right) = |0> 9.16);$$

- Observe that the terms in red cancel (or destruct) each other, which is an example of destructive interference; the terms in blue reinforce (or add to) each other, which is an example of constructive interference. Since interference is a wave phenomenon, and the calculations we have just done are particle-like, we must have run into the wave-particle duality of quantum mechanics.

The Quantum Bit (Qubit)

- A quantum bit (qubit) is any unit vector in a two-dimensional complex vector space spanned by $|0\ >$ and $|1>$. The general state of a qubit $|\varphi>$ is

  $$|\varphi> = c_0|0> + c_1|1> = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \quad \text{Eqn. (9.17).}$$
- The parameters c_0 and c_1 are complex probability amplitudes that satisfy the condition $|c_0|^2 + |c_1|^2 = 1$ as they represent the probability finding the system in state $|0>$ or $|1>$, respectively.
- A single quantum bit is a superposition of two classical bits. But since $|c_0|^2$and $|c_1|^2$ can be of any value, the qubit is represented by a sphere known as the Bloch sphere.



From W. Dur and S. Heusler (2013) https://arxiv.org/abs/1312.1463

Multiple Quantum States

- We introduced the state vectors $|0> = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1> = \begin{bmatrix}0\\1\end{bmatrix}$; these classical bits are components of the quantum bit (qubit) we just defined on the previous slide.

- Just like in the classical case, the dimension of a joint 2-qubit state is the dimension of the composite space.

- The Hilbert space of a composite state is the tensor product of the separate states.

- As in the classical case, the two qubits can be written as

$$\begin{aligned} |00> &= |0> \otimes |0> \\ |01> &= |0> \otimes |1> \\ |10> &= |1> \otimes |0> \\ |11> &= |1> \otimes |1> \end{aligned}$$

- The 2-qubit system lives in a 4-dimensional space.

Superposition of multiple qubits

- As we found out, a single qubit is a superposition of 2 classical states; we can generalize this to the case of the two qubits we discussed in the previous slide.

- If the 4 orthogonal classical states we discussed in the previous slide are normalized, then one can write a general state as a superposition of these classical states, thus

$$|\psi> = c_{00}|00> + c_{01}|01> + c_{10}|10> + c_{10}|11> \text{ where as before,}$$

$$1 = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2.$$

- Following the logic from above, if we have n classical bits, then the generalization to a qubit state is

$$|\psi> = \sum_{0 \le \alpha < 2^n} \beta_\alpha |\alpha>_n$$

Tensors

- For the state vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we can find, as before, the tensor product

$$v \otimes u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 & \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ v_2 & \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_1 u_1 \\ v_1 u_2 \\ v_2 u_1 \\ v_2 u_2 \end{bmatrix}.$$

- If the vectors have numerical values $v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then their tensor product is $v \otimes u = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \\ -6 \\ 2 \\ 3 \end{bmatrix}$

Multiple Qubits

- As in the classical case, for two qubits with state vectors $|\psi>$ and $|\varphi>$ the composite state of the two vectors each in a 2-dimensional space, the tensor is

$$|\Upsilon> = |\psi> \otimes |\varphi>.$$

- And as before, the composite state vector for the 2 qubits has 4 dimensions;

- If we consider n qubits with state vectors $|\varphi_1>, |\varphi_2>, \ldots |\varphi_n>$ each of dimension 2, the composite state will be the tensor

$$|\Phi> = |\varphi_1> \otimes |\varphi_2> \otimes \cdots \otimes |\varphi_n>;$$

- The composite state vector will be of 2^n dimensions.

Playing with Qubits

- Suppose Alice has the quantum state $|\psi> = |+> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and Bob has the quantum state $|\varphi> = |0> = \begin{bmatrix}1\\0\end{bmatrix}$; their combined quantum state is given by the tensor product

$$|\psi> \otimes |\varphi> = |+> \otimes |0> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix};$$

- Alice and Bob have a mutual friend, Charlie, who has the quantum state

$|\chi> = |-> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$; their single combined group quantum state will then be given by

$$|\psi> \otimes |\varphi> \otimes |\chi> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\\1\\-1\\0\\0\end{bmatrix}.$$

Measuring Probabilities of Composite States

- Suppose we have the states $|\psi > = |+1>$ and $|\chi> = |1->$; calculate the probability of measuring the state $|\psi>$ in the state $|\chi>$;
- First, we write the states explicitly as vectors

$$|\psi> = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \text{ and } |\chi> = \begin{bmatrix}0\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix};$$

- By Born's rule, the probability can then be written as

$$P(\chi) = |<\chi|\psi>|^2 = \left|\frac{1}{2}\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right|^2 = \left|-\frac{1}{2}\right|^2 = \frac{1}{4};$$

- Same result can be obtained by writing $P(\chi) = |<+1|1->|^2 = |<+|1>|^2 . |<1|->|^2 = \frac{1}{4}$.

States that Cannot be Written as Tensor Products

- We will stick with 4-dimensional vectors: if we cannot write a 4-dimensional vector as two 2-dimensional vectors, then it means we can't separate the 4-dimensional vector;

- Mathematically, for any 2-dimensional qubits $|\psi >$ and $|\varphi >$, if we have $|\chi > \neq |\psi > |\varphi >$, then we say the state in inseparable.

- On the other hand, if $|\chi > = |\psi > |\varphi >$, then $|\chi >$ is a separable state.

- If a 2-qubit state $|\Psi >$ is written as $|\Psi > = |\varphi > \otimes |\chi >$ for any possible states $|\varphi >$ and $|\chi >$, then the state $|\Psi >$ is entangled.

Quantum Entanglement

- Suppose Alice has the qubit $|A> = c_0|a_0> + c_1|a_1>$ and Bob has the qubit $|B> = d_0|b_0> + d_1|b_1>$, where c_i and d_i are the probability amplitude coefficients.

- These states are defined so that if Alice does a measurement and her state jumps to $|a_0>$, she has a classical 0. Similarly, if Bob performs a measurement and his state jumps to $|b_1>$ he has a classical 1.

- Alice's and Bob's joint state is

$$|AB> = |A> \otimes |B> = (c_0|a_0> + c_1|a_1>) \otimes (d_0|b_0> + d_1|b_1>).$$

- Carrying put the tensor product we get

$$|AB> = c_0 d_0|a_0 b_0> + c_0 d_1|a_0 b_1> + c_1 d_0|a_1 b_0> + c_1 d_1|a_1 b_1>.$$

- If we define $\varrho = c_0 d_0, \varsigma = c_0 d_1, \tau = c_1 d_0, \upsilon = c_1 d_1$, we can write

$$|AB> = \varrho|a_0 b_0> + \varsigma|a_0 b_1> + \tau|a_1 b_0> + \upsilon|a_1 b_1>, \text{ where } |\varrho|^2 + |\varsigma|^2 + |\tau|^2 + |\upsilon|^2 = 1.$$

- In general, when $\varrho\upsilon \neq \varsigma\tau$, Alice's and Bob's states are entangled otherwise they are not when $\varrho\upsilon = \varsigma\tau$.

Joint State Calculations

- Suppose a joint state for Alice and Bob is written as

$$\left|AB >= \frac{\sqrt{3}}{4}\right| a_0 b_0 > + \frac{\sqrt{5}}{4}\left|a_0 b_1 > + \frac{\sqrt{3}}{4}\right| a_1 b_0 > + \frac{\sqrt{5}}{4} |a_1 b_1 >.$$

- To see if the state is entangled or separable, we will factorize it from Alice's perspective, thus

$$|AB >= |a_0 > \left(\frac{\sqrt{3}}{4}\middle|b_0 > + \frac{\sqrt{5}}{4}\middle| b_1 >\right) + |a_1 > \left(\frac{\sqrt{3}}{4}\middle|b_0 > + \frac{\sqrt{5}}{4}\middle| b_1 >\right)$$

- Dividing the expressions in parenthesis by their unit lengths and multiplying outside the parenthesis by their lengths gives us

$$\left|AB >= \frac{1}{\sqrt{2}}\right| a_0 > \left(\frac{\sqrt{6}}{4}\middle|b_0 > + \frac{\sqrt{10}}{4}\middle| b_1 >\right) + \frac{1}{\sqrt{2}}|a_1 > \left(\frac{\sqrt{6}}{4}\middle|b_0 > + \frac{\sqrt{10}}{4}\middle| b_1 >\right)$$

- The state factorizes to $|AB >= \left(\frac{1}{\sqrt{2}}\middle|a_0 > + \frac{1}{\sqrt{2}}\middle| a_1 >\right)\left(\frac{\sqrt{6}}{4}\middle|b_0 > + \frac{\sqrt{10}}{4}\middle| b_1 >\right)$, therefore it is separable.
- We could also have determined the separability by inspecting the product of coefficients of the original state; thus, $\varrho\upsilon = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{\sqrt{5}}{4}\right) = \varsigma\tau = \left(\frac{\sqrt{5}}{4}\right)\left(\frac{\sqrt{3}}{4}\right)$; since they are equal, the state is separable.

Joint State Calculation to Determine Separability

- Suppose the Alice-Bob composite state is now
- $\left|AB > = \frac{1}{4}\right| a_0 b_0 > + \frac{1}{4}\left|a_0 b_1 > + \frac{\sqrt{7}}{2\sqrt{2}}\right| a_1 b_0 > +0|a_1 b_1 >$
- From Alice's perspective we can write the state as

$$|AB > = |a_0 > \left(\frac{1}{4}\left|b_0 > + \frac{1}{4}\right| b_1 >\right) + |a_1 > \left(\frac{\sqrt{7}}{2\sqrt{2}}|b_0 > +0|b_1 >\right)$$

- Normalizing the terms in the parenthesis and diving by the lengths outside the parenthesis, gives

$$\left|AB > = \frac{1}{2\sqrt{2}}\right| a_0 > \left(\frac{1}{\sqrt{2}}\left|b_0 > + \frac{1}{\sqrt{2}}\right| b_1 >\right) + \frac{\sqrt{7}}{2\sqrt{2}}|a_1 > (1|b_1 > +0|b_1 >)$$

- The terms in parenthesis are not the same, which means we cannot factorize the state and it is therefore entangled.
- We could also have determined this by examining the products of the amplitude coefficients of the original state, which are $\varrho\upsilon = 0 \neq \varsigma\tau = \left(\frac{1}{4}\right)\left(\frac{\sqrt{7}}{2\sqrt{2}}\right)$; the state is therefore entangled.

Operations on Qubits

- Quantum computers perform reversible operations on qubits; this means the operations are linear, taking unit vectors to other unit vectors. These operations are called unitary, and the operators are unitary operators defined by

$$uu^{\dagger} = u^{\dagger}u = 1.$$

- For a general $n-$qubit operation, one is performing 2^n $-$dimensional unitary transformation U and like above, we must have

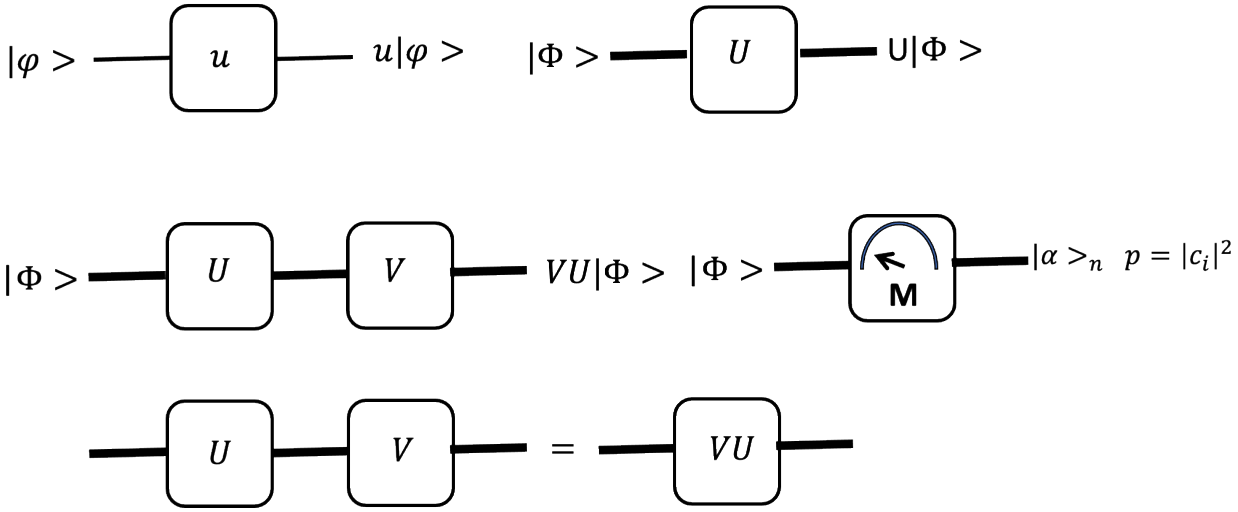
$$UU^{\dagger} = U^{\dagger}U = 1.$$

- We have already encountered the most common unitary quantum operators: they are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ Y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{and}\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- The X is also called the CNOT operator as it negates what it operates on; it is equivalent to the classical NOT gate. The Z operator swaps components of single qubit, I is the identity operator, and H is the Hadamard, which takes a qubit from one basis to another orthonormal basis.

Reversible Operations on Qubits



- Some quantum algorithms are based on unitary operations that act on only one qubit at a time and are built from 1-qubit gates; they can also act on a pair of qubits (2-qubit gates).
- Gate-based quantum computers are constructed from 1- and 2-qubit gates. As in classical computing, any arbitrary transformation can be built up from elementary 1- and 2-qubit gates.
- Illustration of circuit of generic quantum gates in the graphic on the left. Thick lines represent many qubits, and a thin line represents a single qubit. Measurement is represented by M.

Summary

- Introduced the postulates of quantum mechanics relevant for quantum computing
  - Discussed quantum bit as a superposition of two classical bit
  - Classification

- Quantum operations
  - Entanglement
  - Reversible unitary operations on qubits
  - Unitary operators as gates for quantum circuits